

# On Local Description of Two-Dimensional Geodesic Flows with a Polynomial First Integral

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*dedicated to the 55th birthday  
of our friend E.V. Ferapontov*

## Abstract

In this paper we construct multiparametric families of two dimensional metrics with polynomial first integral. Such integrable geodesic flows are described by solutions of some semi-Hamiltonian hydrodynamic type system. We find infinitely many conservation laws and commuting flows for this system. This procedure allows us to present infinitely many particular metrics by the generalized hodograph method.

*keywords:* Geodesic flows, integrability, hydrodynamic type system, Generalized Hodograph Method.

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## 1 Introduction

The problem of integration of geodesic flows on a two-dimensional surface appeared in classical mechanics very early and was extensively investigated in the XIX-th century focusing mostly on local approach. The XX-th century was more centered on global behavior, problems of local description of trajectories were not that much popular. An overview of both aspects may be found in numerous sources, we give here only two references [2, 3]. Partially this loss of interest to the local problem was not only due to the importance of the global problems; it seems reasonable to ascribe this loss of interest to absence of new ideas for the local integrability problem. The situation, in our opinion, has changed in the last decades after publications [5], [1] where the authors had remarked that the equations for the coefficients of first integrals polynomial in momenta for two specific low-dimensional cases belong to the class of diagonalizable hydrodynamic type systems integrable by differential-geometric means; the appropriate theory for such non-linear systems of PDEs was developed in the very end of the XX-th century (cf. [4, 14]). In [12], developing the preliminary results of [5], we demonstrated how to apply the techniques of integrable hydrodynamic type systems to the case of so called one-and-a-half-dimensional systems (one-dimensional mechanical systems with the potential depending on time). Below we investigate (using a bit more sophisticated technologies) the problem of local description of two-dimensional Riemannian metrics with geodesic flows possessing a polynomial first integral of arbitrary high degree.

In [1] the  $N$  component hydrodynamic type system

$$a_t^0 = a^1 a_x^{N-1}, \quad a_t^k = a^{N-1} a_x^{k-1} + [(k+1)a^{k+1} - (N+1-k)a^{k-1}]a_x^{N-1}, \quad (1)$$

where  $k = 1, \dots, N-1$  and  $a^N \equiv 1$ , was derived as the system for the coefficients  $a^k(x, t)$  of a (global smooth) polynomial first integral<sup>1</sup>

$$F(x, t, p_1, p_2) = \sum_{k=0}^N \frac{(-1)^k a^k}{g^{N-k}} p_1^{N-k} p_2^k, \quad (2)$$

for metric in semi-geodesic coordinates

$$ds^2 = g^2(x, t) dt^2 + dx^2 \quad (3)$$

on a 2-dimensional torus, where  $g \equiv a^{N-1}$ . As shown in [1], any smooth metric on a 2-dimensional torus possessing a polynomial first integral can be reduced to such a form. The metric (3) corresponds to the Hamiltonian  $H(x, t, p_1, p_2) = \frac{1}{2}(p_1^2/g^2(x, t) + p_2^2)$ ; the system (1) is equivalent to

$$\{F, H\} = \frac{\partial F}{\partial p_1} \frac{\partial H}{\partial t} - \frac{\partial F}{\partial t} \frac{\partial H}{\partial p_1} + \frac{\partial F}{\partial p_2} \frac{\partial H}{\partial x} - \frac{\partial F}{\partial x} \frac{\partial H}{\partial p_2} = 0.$$

The theorem proved in [1] states that the hydrodynamic type system (1) is diagonalizable (i.e. possesses a complete set of Riemann invariants) and semi-Hamiltonian so may be integrated by the Generalized Hodograph Method (cf. [14]). However the authors of [1] did not give a constructive description of the necessary complete set of hydrodynamic conservation laws and commuting flows for (1).

In this paper we construct  $N$  infinite series of conservation laws and commuting flows. Thus one can construct a rich multiparametric family of particular solutions to (1) by the Generalized Hodograph Method as described below in Sections 2, 7. Our interest is focused on local properties of the system (1) and the respective coefficient  $g(x, t)$  in (3).

The structure of the paper is as follows. In Section 2 we discuss the semi-Hamiltonian property of hydrodynamic type system (1). We construct a generating function of conservation laws and find the equation of the associated Riemann surface. We rewrite hydrodynamic type system (1) in a diagonal form. In Section 3 we rewrite hydrodynamic type system (1) via characteristic velocities and derive analogues of the Löwner equations and the Gibbons–Tsarev system. Also we remark that in the two component case hydrodynamic type system (1) is nothing but the simplest two-component linearly degenerate hydrodynamic type system (which can be written in appropriate field variables in the form  $u_t = v u_x, v_t = u v_x$ ), whose general solution can be presented in implicit form with explicit dependence on two arbitrary functions of a single variable. Such a system has global solutions. This result conforms to the well-known fact that geodesic flows with quadratic first integrals can be integrated by separation of variables. In Section 4 we introduce new field variables  $b^k$  and show how the equation of a Riemann surface rewritten in these field variables  $b^k$  helps in constructing  $N$  infinite series of conservation law

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<sup>1</sup>The factor  $(-1)^k$  was omitted in [1].

densities. In Section 5 we introduce two hydrodynamic chains as infinite sets of equations compatible with hydrodynamic type system (1). This procedure allows us to construct infinitely many conservation laws (the so called Kruskal series) in a compact form. In Section 6 we present the way to construct infinitely many higher commuting flows and infinitely many associated conservation laws. In Section 7 we adopted the Generalized Hodograph Method for construction of a rich infinite-parametric family of particular solutions for the case of hydrodynamic type system (1). Finally in Conclusion (Section 8) we discuss the problem of completeness of the constructed series of conservation laws and briefly expose further perspectives of integrability of two-dimensional geodesic flows.

## 2 Semi-Hamiltonian Systems and their Integration

Integrability of a diagonalizable hydrodynamic type system (similar to (1)) means that such a system possesses  $N$  Riemann invariants  $\mathbf{r} = (r^1, \dots, r^N)$ , has infinitely many hydrodynamic conservation laws and locally any solution can be constructed by the Generalized Hodograph Method:

$$w_i(r^1, \dots, r^N) = v_i(r^1, \dots, r^N) \cdot t + x, \quad i = 1, \dots, N, \quad (4)$$

where  $v_i(\mathbf{r})$  are the characteristic velocities of the system (1) in the diagonal form

$$r_t^i = v_i(\mathbf{r}) r_x^i, \quad i = 1, \dots, N, \quad (5)$$

and  $w_i(\mathbf{r})$  are the coefficients of commuting with (1) flows

$$r_\tau^i = w_i(\mathbf{r}) r_x^i, \quad i = 1, \dots, N, \quad (6)$$

(no summation over repeated indices is assumed anywhere in this paper).

Such systems were called “semi-Hamiltonian” in [14] where the detailed exposition of the respective differential-geometric theory was given.

In this Section we present the equation of a Riemann surface  $\lambda(q, \mathbf{a})$  associated with the hydrodynamic type system (1). Its branch points  $r^i = \lambda|_{q=q_i}$ , where  $q_i$  are solutions of the algebraic equation  $\lambda_q = 0$ , are the Riemann invariants of (1). Also we briefly remind how to find a rich infinite family of conservation laws and commuting flows (6) for (5).

**Theorem [1]:** *Hydrodynamic type system (1) is semi-Hamiltonian.*

In this Section we give an alternative proof of this Theorem obtaining the associated Riemann surface playing the key role in our construction of the explicit formulas for the conservation laws and commuting flows for (1).

**Proof:** According to [1] hydrodynamic type system (1) can be equally derived for the coefficients  $a^k(x, t)$  of a polynomial first integral<sup>2</sup>

$$\tilde{F}(x, t, p) = \sum_{k=0}^N (-1)^k a^k(x, t) p^k (1 - p^2)^{\frac{N-k}{2}} \quad (7)$$

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<sup>2</sup>The corresponding formula in [1] contains misprints.

of Hamilton's equations<sup>3</sup>

$$x' = \frac{\partial \tilde{H}}{\partial p}, \quad p' = -\frac{\partial \tilde{H}}{\partial x}, \quad (8)$$

where the *effective* Hamiltonian function is

$$\tilde{H}(x, t, p) = -a^{N-1}(x, t)\sqrt{1-p^2}.$$

This means that in fact we have a one-dimensional mechanical system with Hamiltonian depending explicitly on the “time”  $t$ . Such systems were called “1.5-dimensional” in [7]. An integrable subclass of 1.5-dimensional systems was studied in [12] where the general technology of hydrodynamic reductions of integrable nonlinear hydrodynamic chains was used for explicit description of various integrable 1.5-dimensional cases. In the present paper we develop a more general approach for the case studied.

In this 1.5-dimensional setting the corresponding Liouville equation<sup>4</sup>

$$f_t = \{f, \tilde{H}\} = \frac{\partial f}{\partial p} \frac{\partial \tilde{H}}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial \tilde{H}}{\partial p}$$

takes the form

$$f_t = a^{N-1}qf_x + (1+q^2)f_q a_x^{N-1}, \quad (9)$$

where instead of the variable  $p$  we introduce the auxiliary variable  $q$  connected with  $p$  via the point transformation

$$q = -\frac{p}{\sqrt{1-p^2}}, \quad p = -\frac{q}{\sqrt{1+q^2}}. \quad (10)$$

Substitution of a general ansatz  $f(x, t, p) = \lambda(q, \mathbf{a}(x, t))$  with a set of some (formally unspecified) field variables  $\mathbf{a} \equiv (a^0(x, t), \dots, a^{N-1}(x, t))$  into (9) leads to an overdetermined compatible systems of equations on  $\partial\lambda/\partial a^i$ ,  $\partial\lambda/\partial q$  if  $a^k(x, t)$  are solutions of hydrodynamic type system (1). Now  $\lambda(q, \mathbf{a}(x, t))$  can be found explicitly:

$$\lambda(q, \mathbf{a}) = (1+q^2)^{-N/2} \left( q^N + \sum_{k=0}^{N-1} q^k a^k \right). \quad (11)$$

In fact (11) can be obtained by a direct substitution of (10) into (7). Thus a Riemann surface with parameters  $(\lambda, q)$  used below for the explicit constructions of the conservation laws and commuting flows is defined. And vice versa, substitution of (11) into (9) and splitting w.r.t.  $q$  leads to (1). In a generic case the algebraic equation  $\lambda_q = 0$  for  $\lambda$  of the form (11), i.e.

$$\sum_{k=0}^{N-1} [(N-k)q^{k+1} - kq^{k-1}]a^k = Nq^{N-1}, \quad (12)$$

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<sup>3</sup>This means that geodesic flows can be written either as stationary Hamilton's equations with two degree of freedom or alternatively as *non-stationary* Hamilton's equations with one-and-a-half degree of freedom. Here the “prime” means the derivative with respect to “ $t$ ”.

<sup>4</sup>The Poisson bracket is standard  $\{f, H\} = f_p H_x - f_x H_p$ . We remind that  $f(x, t, p)$  is a first integral for the given Hamilton's equations, for example  $f = \tilde{F}$ .

has  $N$  simple distinct roots  $q_i = q_i(\mathbf{a})$ . Since  $\lambda_q = 0$  in these points  $q_i = q_i(\mathbf{a}(x, t))$ , Liouville equation (9) leads to the hydrodynamic type system (1), written in the diagonal form

$$r_t^i = a^{N-1} q_i r_x^i, \quad (13)$$

where the Riemann invariants  $r^i = \lambda|_{q=q_i}$  are determined by (11), i.e.

$$r^i(\mathbf{a}) = \lambda(q_i, \mathbf{a}) = (1 + (q_i)^2)^{-N/2} \left( (q_i)^N + \sum_{m=0}^{N-1} (q_i)^m a^m \right), \quad (14)$$

It is easy to see that the constructed  $r^i(\mathbf{a})$  are functionally independent since the characteristic velocities in (13) are distinct. (Functional independence of the velocities  $v_i = a^{N-1} q_i$  will be shown below in Section 3.)

Under the functional inversion transformation  $\lambda = \lambda(q, \mathbf{a}) \rightarrow q = q(\lambda, \mathbf{a})$  the linear equation (9) transforms to

$$q_t = a^{N-1} q q_x - (1 + q^2) a_x^{N-1}, \quad (15)$$

which is equivalent to the equation<sup>5</sup>

$$p_t = \left( a^{N-1} \sqrt{1 - p^2} \right)_x \quad (16)$$

for the so-called generating function of conservation law densities  $p = p(\lambda, a^0, a^1, \dots, a^{N-1})$  where  $\lambda$  is a parameter. Infinitely many conservation laws can be found directly from the equation of Riemann surface (11) expanding the inverse function  $q(\lambda, \mathbf{a})$  (as  $q \rightarrow \infty$ , while  $\lambda \rightarrow 1$ ) w.r.t. the powers of  $(\lambda - 1)$  and substituting into the second relationship of (10) and then into (16).<sup>6</sup> Since coefficients of the resulting expansion of  $p(\lambda, \mathbf{a})$  are conservation law densities (see (16)), and the hydrodynamic type system (1) is diagonalizable (see (13)), we conclude that (1) is semi-Hamiltonian. The Theorem is proved.

The next important step in integration of semi-Hamiltonian system (1) is construction of the necessary complete set of  $w_i(\mathbf{r})$  (depending on  $N$  functions of one variable, cf. [14]) in order to be able to construct any solution of (1) locally using the Generalized Hodograph formula (4). This problem will be discussed in Sections 6, 7. In the following three Sections we consider different forms of the system (1) and the associated Riemann surface (11) in more detail in order to expose the necessary techniques.

### 3 More on Characteristic Velocities and Riemann Invariants. Triviality of the Quadratic Case ( $N = 2$ ).

Since the field variables  $a^k$  in (1) are connected with the characteristic velocities (the eigenvalues of the system matrix) algebraically (see (12) and (13)), one could try to

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<sup>5</sup>Under the potential substitution  $p = S_x$  this equation reduces to the Hamilton–Jacobi equation  $S_t = a^{N-1} \sqrt{1 - (S_x)^2}$  for the mechanical system with the specified Hamiltonian  $H(x, t, p)$ .

<sup>6</sup>This remarkable class of conservation laws is usually called Kruskal series. We describe its construction in more detail below in Section 5.

find explicitly the dependence of the functions  $a^k$  on the characteristic velocities  $v_i(\mathbf{a}) = a^{N-1}q_i(\mathbf{a})$ , if  $v_i(\mathbf{a})$  are functionally independent. Thus automatically the hydrodynamic type system studied in this paper would be written in a symmetric form.

Indeed, (15) yields directly the same hydrodynamic type system (1) written as

$$(q_k)_t = a^{N-1}q_k(q_k)_x - (1 + q_k^2)a_x^{N-1}, \quad (17)$$

where we just replaced  $q$  by the corresponding roots  $q_k$ .

**Remark:** This replacement ( $q \rightarrow q_k$ ) is not so obvious, because roots  $q_k$  of algebraic equation determined by the condition  $\lambda_q = 0$  do not depend on the parameter  $\lambda$ , while the function  $q$  depends on it. In fact  $q_k(\mathbf{a}) = q(\lambda, \mathbf{a})|_{\lambda=r^k(\mathbf{a})}$ . For this reason we give here a precise derivation of the above equations. The general theory of hydrodynamic reductions developed in [6] implies that  $q(\lambda, \mathbf{a}(\mathbf{r}))$  satisfies the overdetermined system (the so called L  wner type equation)

$$\partial_i q = \frac{1 + q^2}{q_i - q} \partial_i A, \quad \partial_i = \partial / \partial r^i, \quad (18)$$

where we denoted  $A = -\ln a^{N-1}$ . These equations are in fact the equation (15) for  $q(\lambda, \mathbf{a}(\mathbf{r}))$  after substitution of (13) and splitting w.r.t.  $r_x^i$ . Compatibility conditions  $\partial_k(\partial_i q) = \partial_i(\partial_k q)$  for (18) yield the so called Gibbons–Tsarev type system

$$\partial_i q_k = \frac{1 + q_k^2}{q_i - q_k} \partial_i A, \quad \partial_i \partial_k A = 2 \frac{1 + q_i q_k}{(q_i - q_k)^2} \partial_i A \partial_k A, \quad i \neq k. \quad (19)$$

One can immediately verify that if  $q_k$  are determined by (19), then (17) is fulfilled. This computation shows that the first (left) part of the Gibbons–Tsarev type system can be formally obtained from (18) by replacement  $q \rightarrow q_k$  as well as (17) from (15).

Taking into account the relationship (see (13)) between characteristic velocities  $v_k = a^{N-1}q_k$  and roots  $q_k$ , the hydrodynamic type system (17) takes the form

$$(v_k)_t = v_k v_x^k + v_k a_x^{N-2} - \frac{2(v_k)^2 + (2a^{N-2} - N)v_k + (a^{N-1})^2}{a^{N-1}} a_x^{N-1}, \quad (20)$$

where  $a^{N-2}$  and  $a^{N-1}$  can be found from (12), written in the form

$$\sum_{k=0}^{N-1} \left( (N-k) \frac{(v_j)^{k+1}}{(a^{N-1})^{k+1}} - k \frac{(v_j)^{k-1}}{(a^{N-1})^{k-1}} \right) a^k = N \frac{(v_j)^{N-1}}{(a^{N-1})^{N-1}}.$$

$j = 1, \dots, N$ . This is a linear algebraic system on unknown functions  $a^k(\mathbf{v})$ ,  $k = 0, \dots, (N-2)$ , except the latest coefficient  $a^{N-1}(\mathbf{v})$ . One can find, for instance, that

$$a^{N-2} = \frac{N}{2} - \frac{1}{2} \sum_{m=1}^N v_m,$$

while  $a^{N-1}(\mathbf{v})$  satisfies different algebraic equations for different  $N$ . For instance, if  $N = 2$ , then  $(a^1)^2 = -V_1$ ; if  $N = 3$ , then

$$(a^2)^2 = \frac{2V_3}{3 - V_1};$$

if  $N = 4$ , then

$$(a^3)^4 + \frac{1}{3}V_2(a^3)^2 + V_4 = 0;$$

if  $N = 5$ , then

$$(a^4)^4 - 2\frac{V_3}{3(5-V_1)}(a^4)^2 - \frac{8V_5}{3(5-V_1)} = 0,$$

etc. Here  $V_k$  are the elementary symmetric polynomials of  $v_j$  of degree  $k$  found by the Vieta Theorem from the formal equation

$$v^k - V_1v^{k-1} + V_2v^{k-2} - V_3v^{k-3} + \dots = 0,$$

so  $V_1 = \sum v_m$ ,  $V_2 = \frac{1}{2}[(\sum v_m)^2 - \sum v_m^2]$ ,  $\dots$

Taking into account (14) we see that all Riemann invariants  $r^k$  can be explicitly expressed via the characteristic velocities  $v_m$ . If  $N = 2$ , we get

$$r^1 = 1 - \frac{1}{2}v_2, \quad r^2 = 1 - \frac{1}{2}v_1, \quad (21)$$

where the corresponding equation of the Riemann surface becomes

$$\lambda(q, \mathbf{v}) = 1 - V_2 \frac{v - \frac{1}{2}V_1}{v^2 - V_2};$$

if  $N = 3$ , then

$$r^k = \frac{(v_k)^3 + \frac{2V_3}{3-V_1}(v_k)^2 + V_3v_k + \frac{1}{3}\frac{2V_3}{3-V_1}\left(V_2 + \frac{4V_3}{3-V_1}\right)}{\left((v_k)^2 + \frac{2V_3}{3-V_1}\right)^{3/2}};$$

if  $N = 4$ , then

$$r^k = 1 + \frac{(a^3)^2(v_k)^3 - \frac{1}{2}V_1(a^3)^2(v_k)^2 - V_4v_k + \frac{1}{12}V_1V_2(a^3)^2 - \frac{1}{4}(a^3)^2V_3 + \frac{1}{4}V_1V_4}{((v_k)^2 + (a^3)^2)^2}.$$

However, inverse formulas for  $v_k(\mathbf{r})$  are much more complicated. Only if  $N = 2$ , the inversion is simple (see (21)):

$$v_2 = 2 - 2r^1, \quad v_1 = 2 - 2r^2.$$

Thus, hydrodynamic type system (1) in the two component case

$$a_t^0 = a^1a_x^1, \quad a_t^1 = a^1a_x^0 + 2(1 - a^0)a_x^1 \quad (22)$$

is linearly degenerate:

$$r_t^1 = 2(1 - r^2)r_x^1, \quad r_t^2 = 2(1 - r^1)r_x^2, \quad (23)$$

where

$$a^0 = r^1 + r^2 - 1, \quad (a^1)^2 = -4(r^1 - 1)(r^2 - 1).$$



**Remark:** Algebraic equation (12) determines  $N$  roots  $q_k(\mathbf{a})$ . However, one can see that inverse expressions  $a^k(\mathbf{q})$  cannot be found if  $N$  is even due to degeneracy of the mapping  $(a^0, \dots, a^{N-1}) \rightarrow (q_1, \dots, q_N)$  for even  $N$ . But if  $N$  is odd, the inverse expressions  $a^k(\mathbf{q})$  can be found. For instance, if  $N = 3$ ,

$$a^1 = \frac{3Q_3}{Q_1 + 2Q_3}, \quad a^0 = \frac{Q_2 + 2}{Q_1 + 2Q_3}, \quad a^2 = \frac{3}{Q_1 + 2Q_3},$$

where  $Q_1 = q_1 + q_2 + q_3$ ,  $Q_2 = q_1q_2 + q_1q_3 + q_2q_3$ ,  $Q_3 = q_1q_2q_3$ . The corresponding expression for the equation of the Riemann surface (11) is

$$\lambda(q, \mathbf{q}) = \frac{(1 + q^2)^{-3/2}}{Q_1 + 2Q_3} [(Q_1 + 2Q_3)q^3 + 3q^2 + 3Q_3q + Q_2 + 2].$$

### 3.1 Linearly Degenerate Case ( $N = 2$ )

Two component hydrodynamic type system (22) corresponding to quadratic first integrals (2) of geodesics on a two-dimensional surface is linearly degenerate (see (23)), that is  $\partial v_1 / \partial r^1 = 0$  and  $\partial v_2 / \partial r^2 = 0$ . Its general solution  $r^1(x, t)$ ,  $r^2(x, t)$  depending on two arbitrary functions  $\beta(u)$  and  $\gamma(v)$  of one variable is well known (cf., for example [13]) and may be presented in implicit form:

$$t = \beta'(u) + \gamma'(v), \quad x = \beta(u) - u\beta'(u) + \gamma(v) - v\gamma'(v),$$

where (for simplicity) we denoted  $u = 2(1 - r^1)$ ,  $v = 2(1 - r^2)$ .

The fact of complete integrability in the case of quadratic first integrals is also well known, its exposition can be found for example in [2] and [3, Ch. 11]. Actually the standard exposition of this case (for the isothermic form  $ds^2 = (f(u) + g(v))(du^2 + dv^2)$  of the metric) in the references given above is equivalent to our result for the metric in semi-geodesic coordinates (3).

## 4 Symmetric Field Variables and $N$ Principal Series of Conservation Laws

In the previous Section the hydrodynamic type system (1) was written in symmetric form (20). However such a symmetric form is non unique. Let us introduce the roots  $b^k$  of the polynomial

$$q^N + \sum_{k=0}^{N-1} q^k a^k = \prod_{k=1}^N (q - b^k),$$

so all field variables  $a^k$  become elementary symmetric polynomials of new field variables  $b^k$ . For instance,

$$a^{N-1} = -\sum_{k=1}^N b^k. \tag{24}$$

Thus, the equation of the Riemann surface (11) takes the form

$$\lambda = (1 + q^2)^{-N/2} \prod_{k=1}^N (q - b^k). \quad (25)$$

After substitution of this expression into (9) the hydrodynamic type system (1) reduces to another simple symmetric form

$$b_t^k = (1 + (b^k)^2) \sum_{m=1}^N b_x^m - \left( \sum_{n=1}^N b^n \right) b^k b_x^k. \quad (26)$$

Indeed, hydrodynamic type system (26) can be derived in three steps:

1. compute the partial derivatives of  $\ln \lambda$  with respect to the independent variables  $x, t, q$ :

$$(\ln \lambda)_x = - \sum_{m=1}^N \frac{b_x^m}{q - b^m}, \quad (\ln \lambda)_t = - \sum_{m=1}^N \frac{b_t^m}{q - b^m}, \quad (\ln \lambda)_q = - \frac{Nq}{1 + q^2} + \sum_{m=1}^N \frac{1}{q - b^m};$$

2. substitution of these derivatives into (9) leads to

$$\sum_{m=1}^N \frac{b_t^m}{q - b^m} = a^{N-1} q \sum_{m=1}^N \frac{b_x^m}{q - b^m} + \left( Nq - \sum_{m=1}^N \frac{q^2 + 1}{q - b^m} \right) a_x^{N-1}$$

or

$$\sum_{m=1}^N \frac{b_t^m}{q - b^m} = a^{N-1} \sum_{m=1}^N b_x^m + a^{N-1} \sum_{m=1}^N \frac{b^m b_x^m}{q - b^m} - \left( \sum_{m=1}^N b^m + \sum_{m=1}^N \frac{(b^m)^2 + 1}{q - b^m} \right) a_x^{N-1};$$

which simplifies to

$$\sum_{m=1}^N \frac{b_t^m + [(b^m)^2 + 1] a_x^{N-1} - a^{N-1} b^m b_x^m}{q - b^m} = a^{N-1} \sum_{m=1}^N b_x^m - \sum_{m=1}^N b^m a_x^{N-1}.$$

3. Taking into account (24) we conclude that the r.h.s. of this equation vanishes so splitting the partial fraction decomposition in the l.h.s. of it w.r.t.  $q$  we get precisely (26).

The variables  $b^k$  are very convenient for explicit computation of commuting flows and conservation laws for the system (1). Since we will need in fact not only the conservation law densities (for example  $p$  appearing in the left hand sides of equations like (16)) but also the fluxes — expressions inside the  $(\dots)_x$  in the right hand sides of such equations — we will call *conservation laws* the equalities (16) themselves.

Now we are ready to explain our method of deriving explicit formulas for conservation laws of (1) using the Riemann surface associated to this integrable hydrodynamic type system. One should note that  $N$  component semi-Hamiltonian hydrodynamic type system has infinitely many conservation laws and commuting flows — both families are parameterized by  $N$  arbitrary functions of a single variable (see details in [14]). In many

interesting cases this functional dependence cannot be presented in explicit form. Nevertheless  $N$  infinite series of conservation laws and commuting flows can be constructed, for instance, if the corresponding equation of associated Riemann surface is known. As one can prove (cf. for example [11]) for a vast class of semi-Hamiltonian systems such series form the complete basis for the (infinite-dimensional) linear space of all conservation laws (or commuting flows). Another techniques for proving completeness can be found in [12].

$N$  **infinite series of conservation laws for (26)** can be found in three steps:

1. Let us expand<sup>7</sup>  $q$  with respect to the local parameter  $\lambda$  at the vicinity of each root  $b^k$ :

$$q^{(k)}(\lambda) = b^k + \lambda q_1^{(k)} + \lambda^2 q_2^{(k)} + \lambda^3 q_3^{(k)} + \dots, \quad \lambda \rightarrow 0. \quad (27)$$

All coefficients  $q_m^{(k)}$  can be found recursively, for instance

$$q_1^{(k)} = \frac{(1 + (b^k)^2)^{N/2}}{\prod_{m \neq k} (b^k - b^m)}.$$

2. Substitute the series  $q^{(k)}(\lambda)$  into (10), obtaining expansion

$$p^{(k)}(\lambda) = -\frac{q^{(k)}(\lambda)}{\sqrt{1 + (q^{(k)}(\lambda))^2}} = h^k + \lambda p_1^{(k)} + \lambda^2 p_2^{(k)} + \lambda^3 p_3^{(k)} + \dots, \quad (28)$$

where

$$h^k = -\frac{b^k}{\sqrt{1 + (b^k)^2}}. \quad (29)$$

Then again all conservation law densities  $p_m^{(k)}$  can be found recursively, for instance

$$p_1^{(k)} = -\frac{(1 + (b^k)^2)^{(N-3)/2}}{\prod_{m \neq k} (b^k - b^m)}. \quad (30)$$

3. Substitute  $p^{(k)}(\lambda)$  into the generating function  $p$  of conservation laws (16)

$$(p^{(k)}(\lambda))_t + \left( \sqrt{1 - (p^{(k)}(\lambda))^2} \sum_{n=1}^N b^n \right)_x = 0$$

and expand both sides w.r.t. the parameter  $\lambda \rightarrow 0$ . Matching the coefficients of the same powers of  $\lambda$  one obtains  $N$  infinite series of conservation laws, namely

$$(h^k)_t + \left( \sqrt{1 - (h^k)^2} \sum_{n=1}^N b^n \right)_x = 0, \quad (p_1^{(k)})_t = \left( \frac{h^k p_1^{(k)}}{\sqrt{1 - (h^k)^2}} \sum_{n=1}^N b^n \right)_x, \dots \quad (31)$$

---

<sup>7</sup>This expansion is given by the so called Lagrange–Bürmann inversion (see [12] for an example of its use).

**Remark:** As a by-product we obtain the hydrodynamic type system (1), (26) in a symmetric conservative form

$$(h^k)_t = \left( \sqrt{1 - (h^k)^2} \sum_{m=1}^N \frac{h^m}{\sqrt{1 - (h^m)^2}} \right)_x, \quad k = 1, \dots, N, \quad (32)$$

where we utilized the inverse point transformation (cf. (10), (29))

$$b^k = -\frac{h^k}{\sqrt{1 - (h^k)^2}}.$$

Its first  $N$  conservation laws are (31) expressed in variables  $h^k$  using (29).

Alongside with the constructed  $N$  series of conservation laws one can obtain another (incomplete) set of conservation laws usually called Kruskal series. Its form is much simpler and is symmetric w.r.t. the variables  $b^i$ . We give Kruskal series below in Section 5.

## 5 Hydrodynamic Chains and Kruskal Series of Conservation Laws

If we introduce new variables (called moments)

$$B^k = \frac{1}{k+1} \sum_{m=1}^N (b^m)^{k+1}, \quad k = 0, 1, \dots, \quad (33)$$

then hydrodynamic type system (26) implies infinitely many equations

$$B_t^0 = (N + 2B^1)B_x^0 - B^0 B_x^1, \quad B_t^k = (kB^{k-1} + (k+2)B^{k+1})B_x^0 - B^0 B_x^{k+1}, \quad k = 1, 2, \dots; \quad (34)$$

just first  $N$  of them are independent since due to (33) all higher moments  $B^N, B^{N+1}, \dots$  are polynomial expressions of the first  $N$  moments so hydrodynamic type system (26) in moments  $B^k$  takes the form

$$B_t^0 = (N + 2B^1)B_x^0 - B^0 B_x^1, \quad B_t^k = (kB^{k-1} + (k+2)B^{k+1})B_x^0 - B^0 B_x^{k+1}, \quad k = 1, 2, \dots, N-2, \quad (35)$$

$$B_t^{N-1} = [(N-1)B^{N-2} + (N+1)B^N(\mathbf{B})]B_x^0 - B^0(B^N(\mathbf{B}))_x,$$

where the dependence  $B^N(\mathbf{B}) = B^N(B^0, \dots, B^{N-1})$  is to be found from (33) using the standard combinatorial results on symmetric polynomials. For instance,

1. if  $N = 2$ , then

$$B^2(\mathbf{B}) = B^0 B^1 - \frac{1}{6}(B^0)^3;$$

2. if  $N = 3$ , then

$$B^3(\mathbf{B}) = B^0 B^2 + \frac{1}{2}(B^1)^2 - \frac{1}{2}(B^0)^2 B^1 + \frac{1}{24}(B^0)^4;$$

3. if  $N = 4$ , then

$$B^4(\mathbf{B}) = B^0 B^3 + B^1 B^2 - \frac{1}{2}(B^0)^2 B^2 - \frac{1}{2}B^0(B^1)^2 + \frac{1}{6}(B^0)^3 B^1 - \frac{1}{120}(B^0)^5, \dots$$

Nevertheless, one can consider infinitely many equations (34) without above restrictions. Similar infinite chains of quasilinear first-order equations (called hydrodynamic chains) are very useful for integration of various semi-Hamiltonian systems appearing in applications. An overview of this approach can be found in [10, 12].

The associated Riemann surface (25) can be expanded at infinity as  $q \rightarrow \infty$  and  $\lambda \rightarrow 1$ . For convenience we replace below  $\lambda$  by  $\mu = -\ln \lambda$ , so  $\mu \rightarrow 0$  and

$$\mu = \frac{N}{2} \ln(1 + q^2) - \sum_{k=1}^N \ln(q - b^k).$$

Then (see (33)) asymptotically

$$\mu = \frac{N}{2} \ln \left( 1 + \frac{1}{q^2} \right) + \sum_{k=0}^{\infty} \frac{B^k}{q^{k+1}} = \sum_{k=0}^{\infty} \frac{C^k}{q^{k+1}}, \quad q \rightarrow \infty, \quad (36)$$

and introducing new field variables  $C^k$  using their relation to  $B^k$  in (36) we obtain another remarkable hydrodynamic chain

$$C_t^k = (kC^{k-1} + (k+2)C^{k+1})C_x^0 - C^0 C_x^{k+1}, \quad k = 0, 1, 2, \dots, \quad (37)$$

where  $C^{2k} = B^{2k}$  and  $C^{2k-1} = B^{2k-1} - (-1)^k \frac{N}{2k}$ .

Embedding of hydrodynamic type system (26) into the hydrodynamic chain (37) for arbitrary  $N$  allows us to find Kruskal conservation laws<sup>8</sup> in a compact form. First we (using the Lagrange–Bürmann inversion) get from (36) the asymptotic decomposition of  $q$  and  $p$  as  $\mu \rightarrow 0$ :

$$q(\mu) = \frac{C^0}{\mu} + \frac{C^1}{C^0} + \mu \left( \frac{C^2}{(C^0)^2} - \frac{(C^1)^2}{(C^0)^3} \right) + \mu^2 \left( \frac{C^3}{(C^0)^3} - \frac{3C^1 C^2}{(C^0)^4} + \frac{2(C^1)^3}{(C^0)^5} \right) + \dots, \quad (38)$$

$$p(\mu) = -1 + \frac{\mu^2}{2(C^0)^2} - \mu^3 \frac{C^1}{(C^0)^4} + \mu^4 \left( -\frac{C^2}{(C^0)^5} + \frac{5(C^1)^2}{2(C^0)^6} - \frac{3}{8(C^0)^4} \right) + \dots \quad (39)$$

Substitute the last expansion into (16), note that  $a^{N-1} = -C^0$  and equate the coefficients at equal powers of  $\mu$  obtaining the Kruskal series of conservation laws. The first few of them are:

$$((C^0)^{-2})_t = [2C^1(C^0)^{-2}]_x, \quad [C^1(C^0)^{-4}]_t = \left( \frac{2(C^1)^2}{(C^0)^4} - \frac{C^2}{(C^0)^3} - \frac{1}{2(C^0)^2} \right)_x, \quad (40)$$

$$\left( \frac{C^2}{(C^0)^5} - \frac{5(C^1)^2}{2(C^0)^6} + \frac{3}{8(C^0)^4} \right)_t = \left( \frac{3C^1}{2(C^0)^4} + \frac{5C^2 C^1}{(C^0)^5} - \frac{5(C^1)^3}{(C^0)^6} - \frac{C^3}{(C^0)^4} \right)_x.$$

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<sup>8</sup>We call this asymptotic expansion Kruskal, because M. Kruskal was the first who introduced a similar construction for the KdV equation.

## 6 Commuting Flows

The most important part of the integration procedure for a semi-Hamiltonian system (as described briefly in Section 2) is construction (preferably in an explicit form) of sufficiently many commuting flows, either in a diagonal form (6) or in non-diagonal representation in terms of the variables  $a^k$  or  $b^k$ . We choose the latter possibility in this Section; the next Section is devoted to a suitable modification of the Generalized Hodograph formula (4) to the non-diagonal form of the commuting flows.

In the case considered in this paper the conservation laws for the generating function have the form (see [8])

$$p_t = (S(p, b^1, b^2, \dots, b^N))_x$$

where the flux  $S(p, b^1, b^2, \dots, b^N)$  can be represented in a much simpler form  $S(p, B^0)$  (cf. [9]), where  $B^0(\mathbf{b})$  is the “zeroth” moment (of the corresponding hydrodynamic chain (34)). In this paper we consider the case  $S = S_1(p, B^0) = -B^0\sqrt{1-p^2}$ . The situation with higher commuting flows for the chain (34) is as follows: the first commuting flow has the flux (corresponding to the same generating function  $p(\mathbf{b}, \lambda)$ ) of the form  $S_2(p, B^0, B^1)$ , the second commuting flow has the flux  $S_3(p, B^0, B^1, B^2)$  etc. Assume that all these functions  $S_1(p, B^0)$ ,  $S_2(p, B^0, B^1)$ ,  $S_3(p, B^0, B^1, B^2)$ ,  $\dots$  are the coefficients of an expansion of some more complicated function w.r.t. an extra parameter  $\zeta$ , we come to the ansatz that such a function should be written also in a compact form  $G(p(\lambda), p(\zeta))$  (see a more general exposition in [10]).

The conservation laws for the generating function of all higher commuting flows were found in [10]. For this particular case they can be represented formally as

$$\partial_{\tau(\eta)} p(\mu) = \partial_x G(p(\mu), p(\eta)), \quad (41)$$

where  $\mu = -\ln \lambda$ ,  $p(\eta)$  is obtained by formally replacing the parameter  $\mu$  by the parameter  $\eta$  and

$$G(p(\mu), p(\eta)) = \frac{\sqrt{1-p^2(\mu)}}{\sqrt{1-p^2(\eta)}} + \frac{1}{2}p(\mu) \ln \frac{p(\eta)+1}{p(\eta)-1} + \ln \frac{p(\mu)-p(\eta)}{\sqrt{1-p^2(\mu)} + \sqrt{1-p^2(\eta)}}. \quad (42)$$

The so called “vertex” operator  $\partial_{\tau(\eta)}$  is not yet determined and should be specified separately for different cases below.

### 6.1 The Kruskal Series of Commuting Flows

For this series infinitely many respective fluxes for the generating function of conservation laws of the higher commuting flows can be found substituting the following asymptotic expansions for  $\eta \rightarrow 0$  (cf. (38) and (39))

$$q(\eta) = \frac{C^0}{\eta} + \frac{C^1}{C^0} + \eta \left( \frac{C^2}{(C^0)^2} - \frac{(C^1)^2}{(C^0)^3} \right) + \eta^2 \left( \frac{C^3}{(C^0)^3} - \frac{3C^1C^2}{(C^0)^4} + \frac{2(C^1)^3}{(C^0)^5} \right) + \dots, \quad (43)$$

$$p(\eta) = -1 + \frac{\eta^2}{2(C^0)^2} - \eta^3 \frac{C^1}{(C^0)^4} + \eta^4 \left( -\frac{C^2}{(C^0)^5} + \frac{5(C^1)^2}{5(C^0)^6} - \frac{3}{8(C^0)^4} \right) + \dots \quad (44)$$

into (42) and (41) and specifying the expansion of the vertex operator

$$\partial_{\tau(\eta)} = \ln \eta \partial_{t^0} + \frac{1}{\eta} \partial_{t^1} + \partial_{t^2} + \eta \partial_{t^3} + \eta^2 \partial_{t^4} + \dots$$

to match with the expansion

$$\begin{aligned} G(p(\mu), p(\eta)) &= p(\mu) \ln \eta + \frac{C^0 \sqrt{1 - p^2(\mu)}}{\eta} \\ &+ \left[ \frac{C^1}{C^0} \sqrt{1 - p^2(\mu)} - p(\mu) \ln C^0 + \frac{1}{2} \ln \frac{p(\mu) + 1}{p(\mu) - 1} - p(\mu) \ln 2 \right] \\ &+ \eta \left[ \left( \frac{C^2}{(C^0)^2} - \frac{(C^1)^2}{(C^0)^3} + \frac{1}{2C^0} \right) \sqrt{1 - p^2(\mu)} - \frac{C^1}{(C^0)^2} p(\mu) - \frac{1}{C^0 \sqrt{1 - p^2(\mu)}} \right] + \dots \end{aligned}$$

So now we can identify  $t^0 \equiv x$  and  $t^1 \equiv t$ , so

$$(p(\mu))_{t^0} = (p(\mu))_x, \quad (p(\mu))_{t^1} = - \left( C^0 \sqrt{1 - p^2(\mu)} \right)_x,$$

while higher conservation laws are (cf. (16) and (24))

$$\begin{aligned} (p(\mu))_{t^2} &= \left( \frac{C^1}{C^0} \sqrt{1 - p^2(\mu)} - p(\mu) \ln C^0 + \frac{1}{2} \ln \frac{p(\mu) + 1}{p(\mu) - 1} - p(\mu) \ln 2 \right)_x, \\ (p(\mu))_{t^3} &= \left[ \left( \frac{C^2}{(C^0)^2} - \frac{(C^1)^2}{(C^0)^3} + \frac{1}{2C^0} \right) \sqrt{1 - p^2(\mu)} - \frac{C^1}{(C^0)^2} p(\mu) - \frac{1}{C^0 \sqrt{1 - p^2(\mu)}} \right]_x, \dots \end{aligned}$$

Corresponding higher Liouville equations are associated with higher commuting hydrodynamic chains. For instance, the first higher Liouville equation

$$f_{t^2} = \left( q^2 + \frac{C^1}{C^0} q - \ln C^0 + 1 - \ln 2 \right) f_x + (q^2 + 1) f_q \left( q \ln C^0 + \frac{C^1}{C^0} \right)_x$$

is associated with the first higher hydrodynamic chain commuting with (37):

$$\begin{aligned} C_{t^2}^k &= C_x^{k+2} + \frac{C^1}{C^0} C_x^{k+1} - (\ln C^0 - 1 + \ln 2) C_x^k - [(k+2)C^{k+1} + kC^{k-1}] \frac{C_x^1}{C^0} \\ &+ [C^1(kC^{k-1} + (k+2)C^{k+1}) - C^0((k+1)C^k + (k+3)C^{k+2})] \frac{C_x^0}{(C^0)^2}. \end{aligned}$$

## 6.2 $N$ Principal Series of Commuting Flows

As we remarked above any  $N$  component semi-Hamiltonian hydrodynamic type system has infinitely many conservation laws and commuting flows parameterized by  $N$  arbitrary functions of a single variable (see detail in [14]), but in many interesting cases only  $N$  infinite series of conservation laws and commuting flows can be constructed (cf. [11, 12,

14]); they usually form a complete basis in the respective linear spaces. Completeness of  $N$  series constructed here is discussed in Conclusion.

In this case we again (as in the previous sub-Section) start from (41) and (42).

The fluxes for the generating function of conservation laws of corresponding higher commuting flows can be found from the equations written in a conservative form (cf. (32))

$$(h_i)_{\tau(\zeta)} = \partial_x G(h_i, p(\zeta)). \quad (45)$$

Substitution (see (28)) of the expansion of  $p(\zeta)$  as  $\zeta \rightarrow 0$  leads to construction of higher commuting flows. However, this derivation is not so trivial. Indeed, corresponding expansion of the “vertex” operator  $\partial_{\tau(\zeta)}$  matching the obtained below expansion of the r.h.s. of (45) is simple:

$$\partial_{\tau^{(k)}(\zeta)} = \partial_{t^{0,k}} + \zeta \partial_{t^{1,k}} + \zeta^2 \partial_{t^{2,k}} + \zeta^3 \partial_{t^{3,k}} + \dots, \quad (46)$$

where the index  $k = 1, \dots, N$  in  $\tau^{(k)}(\zeta)$  means  $k$ -th branch of Riemann surface (25). However, direct substitution of (28) and (46) yields the desirable infinite set of equations

$$(h^i)_{t^{0,k}} = \partial_x G(h^i, h^k), \quad (h^i)_{t^{1,k}} = \partial_x G_1(h^i, h^k, p_1^{(k)}), \quad (h^i)_{t^{2,k}} = \partial_x G_2(h^i, h^k, p_1^{(k)}, p_2^{(k)}), \dots$$

only for *distinct* indices  $i$  and  $k$ , because the function  $G(x, y)$  has a singularity (see (42)) for  $x = y$ :

$$G(x, y) = Q(x, y) + \ln(x - y), \quad (47)$$

with the non-singular part

$$Q(x, y) = \frac{\sqrt{1-x^2}}{\sqrt{1-y^2}} + \frac{1}{2}x \ln \frac{y+1}{y-1} - \ln \left( \sqrt{1-x^2} + \sqrt{1-y^2} \right).$$

To complete the construction for  $i = k$ , one should observe that substitution of (28) into (47) leads to the asymptotic expansion ( $\zeta \rightarrow 0$ )

$$G(h^k, p^{(k)}(\zeta)) = Q(h^k, h^k + \zeta p_1^{(k)} + \zeta^2 p_2^{(k)} + \zeta^3 p_3^{(k)} + \dots) + \ln(p_1^{(k)} + \zeta p_2^{(k)} + \zeta^2 p_3^{(k)} + \dots) + \ln \zeta.$$

Since the leading term  $\ln \zeta$  disappears in (45) after differentiation, we obtain the necessary matching expansions for  $G$  and  $\partial_{\tau(\zeta)}$  for  $i = k$ . For instance,

$$(h^k)_{t^{0,k}} = [Q(h^k, h^k) + \ln p_1^{(k)}]_x, \quad (h^k)_{t^{1,k}} = \left( p_1^{(k)} \frac{\partial Q(h^k, p)}{\partial p} \Big|_{p=h^k} + \frac{p_2^{(k)}}{p_1^{(k)}} \right)_x, \dots$$

Conservation law densities  $p_m^{(k)}$  were described in Section 4. Thus,  $N$  first commuting flows are

$$(h^i)_{t^{0,k}} = \left( \frac{\sqrt{1-(h^i)^2}}{\sqrt{1-(h^k)^2}} + \frac{1}{2}h^i \ln \frac{h^k+1}{h^k-1} + \ln \frac{h^i-h^k}{\sqrt{1-(h^i)^2} + \sqrt{1-(h^k)^2}} \right)_x, \quad i \neq k;$$

$$(h^k)_{t^{0,k}} = \left[ \frac{1}{2}h^k \ln \frac{h^k+1}{h^k-1} + \ln \frac{\sqrt{1-(h^k)^2} \prod_{m \neq k} \sqrt{1-(h^m)^2}}{\prod_{m \neq k} (h^k \sqrt{1-(h^m)^2} - h^m \sqrt{1-(h^k)^2})} \right]_x.$$



## 7 Generalized Hodograph Method

In this Section we finish our procedure for integration of hydrodynamic type system (1), also written in equivalent forms (20), (26), (32), (35).

According to the Generalized Hodograph Method (see detail in [14]), any generic solution  $r^i(x, t)$  of a semi-Hamiltonian diagonal hydrodynamic type system (5) in a neighborhood of a generic point is given in an implicit form by the algebraic system (4) for the unknowns  $r^i(x, t)$ , where  $w^i(\mathbf{r})$  are the velocities of a generic commuting flow (6). In arbitrary hydrodynamic variables  $u^i(\mathbf{r})$  one can easily rewrite the algebraic system (4) (cf. [14]) as

$$x\delta_k^i - tv_j^i(\mathbf{u}) = w_j^i(\mathbf{u}), \quad (48)$$

where the hydrodynamic type system (5) has the form

$$u_t^i = \sum_j v_j^i(\mathbf{u})u_x^j, \quad i, j = 1, \dots, N,$$

while commuting hydrodynamic type systems (6) have the form

$$u_\tau^i = \sum_j w_j^i(\mathbf{u})u_x^j, \quad i, j = 1, \dots, N.$$

In this Section we will modify (4), (48) further in order to get the simplest form suitable for the case studied.

In order to construct solutions of (26) we first need to prove the following result suitable for our particular case of system (1) (cf. [11, 12]):

**Lemma:** *Hydrodynamic type system (26) together with commuting flows (45) has the common conservation law*

$$dz = \frac{1}{(C^0)^2}dx + \frac{2C^1}{(C^0)^2}dt + \left( \frac{1}{2(C^0)^2} \ln \frac{p(\eta) + 1}{p(\eta) - 1} + \frac{p(\eta)}{(C^0)^2(1 - p^2(\eta))} \right) d\tau(\eta).$$

**Proof:** If  $\mu \rightarrow 0$ ,

$$G(p(\mu), p(\eta)) = \mu^2 \left( \frac{1}{4(C^0)^2} \ln \frac{p(\eta) + 1}{p(\eta) - 1} + \frac{p(\eta)}{2(C^0)^2(1 - p^2(\eta))} \right) + \mu^3(\dots) + \dots$$

Taking into account (41) and (44), one can obtain

$$\left( \frac{1}{(C^0)^2} \right)_{\tau(\eta)} = \left( \frac{1}{2(C^0)^2} \ln \frac{p(\eta) + 1}{p(\eta) - 1} + \frac{p(\eta)}{(C^0)^2(1 - p^2(\eta))} \right)_x.$$

This conservation law together with (40) can be written in the above potential form, where  $z$  is a potential function such that  $z_x = (C^0)^{-2}$ . The Lemma is proved.

Next we modify the initial Generalized Hodograph formula (4). It should be reduced (following the idea in [12]) to a form suitable for our non-diagonal systems (4) and (26).

Algebraic system (4), (or (48) in arbitrary variables) can be written in the form (here  $\partial_i = \partial/\partial r^i$ )

$$x + t \frac{\partial_i G_1}{\partial_i H_0} = \frac{\partial_i G_\infty}{\partial_i H_0}, \quad (49)$$

where we denoted  $H_0 = (C^0)^{-2}$ ,  $G_1 = 2C^1(C^0)^{-2}$  and

$$G_\infty = \frac{1}{2(C^0)^2} \ln \frac{p(\lambda) + 1}{p(\lambda) - 1} + \frac{p(\lambda)}{(C^0)^2(1 - p^2(\lambda))}.$$

Here we returned to the original parameter  $\lambda$  to indicate that now we are working not with any particular asymptotic expansion (cf. (28), (44)), but with the original algebraic surface as a whole. Indeed, hydrodynamic type system (5) has a conservation law  $\partial_t H_0 = \partial_x G_1$ , while the commuting hydrodynamic system has the conservation law  $\partial_\tau H_0 = \partial_x G_\infty$ . This means that  $\sum \partial_i H_0 \cdot r_t^i = \sum \partial_i G_1 \cdot r_x^i$  and  $\sum \partial_i H_0 \cdot r_\tau^i = \sum \partial_i G_\infty \cdot r_x^i$ . Taking into account (5), (6), (4) and splitting w.r.t.  $r_x^i$ , one obtains (49). Multiplying (49) by  $\partial_i H_0 \cdot dr^i$  and summing up, one arrives at

$$xdH_0(\mathbf{r}) + tdG_1(\mathbf{r}) = dG_\infty(\mathbf{r}).$$

Now we rewrite this equation after the invertible point transformation  $(\mathbf{r}) \rightarrow (\mathbf{b})$  as  $xdH_0(\mathbf{b}) + tdG_1(\mathbf{b}) = dG_\infty(\mathbf{b})$ , so the algebraic system (4) becomes

$$x \frac{\partial H_0}{\partial b^i} + t \frac{\partial G_1}{\partial b^i} = \frac{\partial G_\infty}{\partial b^i}.$$

Taking into account  $\partial_i H_0 = -2(C^0)^{-3}$ ,  $\partial_i G_1 = 2(C^0)^{-2}b^i - 4C^1(C^0)^{-3}$  (see (33), here  $\partial_i = \partial/\partial b^i$ ), we obtain the algebraic system

$$x + t(2C^1 - C^0 b^i) = \ln \left( \sqrt{1 + q^2} - q \right) - q\sqrt{1 + q^2} + C^0 \frac{\sqrt{1 + q^2}}{q - b^i} \left( \sum_{m=1}^N \frac{1}{q - b^m} - \frac{Nq}{1 + q^2} \right)^{-1}, \quad (50)$$

where  $q(\mathbf{b}, \lambda)$  is the inverse function to the function  $\lambda(\mathbf{b}, q)$  (25). These  $N$  equations are nothing but the diagonal part of the matrix algebraic system (48). All off-diagonal equations are compatible with the diagonal part ([14]).

So we proved:

**Theorem 1** *Hydrodynamic type system (26) has infinitely many particular solutions  $b^i(x, t)$  in the implicit form given by (50) with a free parameter  $\lambda$ .*

The algebraic system (50) determines one parametric family of solutions  $b^i(x, t, \lambda)$  in implicit form and simultaneously  $g(x, t, \lambda) = -C^0(\mathbf{b}(x, t, \lambda))$ . Thus we found one parametric family of Hamilton's equations (8), which are Liouville integrable.

In fact, using the Generalized Hodograph Method and the *nonlinear superposition principle* implied by this method (see below) we easily obtain multiparametric families of integrable metrics. Namely expanding the generating function  $p(\mathbf{b}, \lambda)$  at different points on the Riemann surface  $p = p(\mathbf{b}, \lambda)$  with the parameters  $(p, \lambda)$  (for example when  $p \rightarrow -1$  or  $p \rightarrow b^i$ ), one can construct infinite multiparametric series of new solutions  $g(\mathbf{b}(x, t))$ . Now we demonstrate this idea. To avoid large repeated expressions, we introduce functions

$$W_i(\mathbf{b}, \lambda) = \ln \left( \sqrt{1 + q^2} - q \right) - q\sqrt{1 + q^2} + C^0 \frac{\sqrt{1 + q^2}}{q - b^i} \left( \sum_{m=1}^N \frac{1}{q - b^m} - \frac{Nq}{1 + q^2} \right)^{-1}.$$

1. *Kruskal series.* Substitution of asymptotic expansion (43) into (50) leads to

$$W_i(\mathbf{b}, \mu) = \ln \mu + \frac{1}{\mu} W_i^{(-1)}(\mathbf{b}) + W_i^{(0)}(\mathbf{b}) + \mu W_i^{(1)}(\mathbf{b}) + \mu^2 W_i^{(2)}(\mathbf{b}) + \dots,$$

where, for instance,

$$\begin{aligned} W_i^{(-1)}(\mathbf{b}) &= C^0 b^i - 2C^1. \\ W_i^{(0)}(\mathbf{b}) &= (b^i)^2 - \frac{C^1}{C^0} b^i + \frac{2(C^1)^2 - 3C^0 C^2}{(C^0)^2} - \log C^0 - \log 2. \end{aligned}$$

2. *N principal series.* Substitution of asymptotic series (27) into (50) leads to

$$W_i^{(k)}(\mathbf{b}, \lambda) = W_{i0}^{(k)}(\mathbf{b}) + \lambda W_{i1}^{(k)}(\mathbf{b}) + \lambda^2 W_{i2}^{(k)}(\mathbf{b}) + \dots,$$

where, for instance,

$$W_{i0}^{(k)}(\mathbf{b}) = \ln \left( \sqrt{1 + (b^k)^2} - b^k \right) + (C^0 \delta_i^k - b^k) \sqrt{1 + (b^k)^2}.$$

Once we found all these coefficients  $W_{im}^k(\mathbf{b})$  and the Kruskal series  $W_i^k(\mathbf{b})$ , we can construct infinitely many particular solutions parameterized by arbitrary number of constants  $\sigma_k^m$ :

$$x + t(2C^1 - C^0 b^i) = \sum_{k=1}^N \sum_{m=0}^{\infty} \sigma_k^m W_{im}^k(\mathbf{b}), \quad (51)$$

or a functional parameter  $\varphi(\lambda)$ :

$$x + t(2C^1 - C^0 b^i) = \oint \varphi(\lambda) W_i(\mathbf{b}, \lambda) d\lambda, \quad (52)$$

where  $\varphi(\lambda)$  and the contour can be chosen in many special forms. Formulae (51) and (52) present the *nonlinear superposition principle* implied by the Generalized Hodograph Method.

## 8 Conclusion

In this paper we considered integrability of semi-Hamiltonian hydrodynamic type system (1). We found and presented  $N$  infinite (principal) series of conservation laws and commuting flows. This means that one can extract infinitely many corresponding solutions by the Generalized Hodograph Method. Here we did not investigate a completeness of conservation law densities (and correspondingly commuting flows). This problem should be investigated elsewhere. Here we just mention that (see (30)) for  $N = 2n + 1$ ,  $n > 0$

$$\sum_{k=1}^{2n+1} p_1^{(k)} = - \frac{(1 + (b^k)^2)^{n-1}}{\prod_{m \neq k} (b^k - b^m)} = 0.$$

This means that corresponding conservation law densities  $p_1^{(k)}$  are not linearly independent for odd  $N$ . Thus one probably should construct additional conservation law densities to get the complete set.

Semi-Hamiltonian hydrodynamic type system (1) holds for the coefficients  $a^k(x, t)$  of a first (polynomial) integral (2). In this paper we found more appropriate set of field variables  $b^k$ . This natural choice of unknown functions follows also from factorization of polynomial expression for the first integral with respect to ratio of both momenta<sup>9</sup>:

$$F = \sum_{k=0}^N \frac{(-1)^k a^k}{g^{N-k}} p_1^{N-k} p_2^k = \left( \frac{p_1}{g} \right)^N \prod_{k=1}^N (q - b^k).$$

Also this first integral can be written in two other equivalent forms (see (33) and (36)):

$$F = (p_2)^N \exp \left( - \sum_{k=0}^{\infty} \frac{B^k}{q^{k+1}} \right) = \left[ (p_2)^2 + \left( \frac{p_1}{B^0} \right)^2 \right]^{N/2} \lambda(q, \mathbf{B}),$$

where

$$q = \frac{B^0 p_2}{p_1}.$$

These two equivalent representations for the first integral are of big interest: if one finds another finite-dimensional parameterization of the moments  $B^k(\mathbf{b})$  compatible with (34), the first integral associated with this solution will be no longer polynomial. This more general problem should be considered in a separate publication.

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<sup>9</sup>we remind that  $a^N = 1$  and  $a^{N-1} = g$ .

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